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LANCHESTER MODELS
OF GUERRILLA ENGAGEMENTS

Marvin B. Schaffer

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PREFACE

This Memorandum develops a set of Lanchester-type equations modeling small-force guerrilla engagements that are typical of the early stages of insurgency. These equations include the effects of supporting weapons and the discipline or morale of the troops involved. The novelty of this treatment from the mathematical point of view is the use of time-dependent weapon-efficiency coefficients.

Although the models and theory are not adequate to predict the outcome of an insurgency, they should prove useful for examining the credibility of casualty claims associated with such conflict. In addition, the models provide useful insight regarding the important attack parameters of guerrilla warfare. In some cases, notably those concerning ambushes, the theory suggests new military hardware which in the past has been difficult to justify on analytical grounds.

This study was undertaken for the Advanced Research Projects Agency Remote Area Conflict Project (Project AGILE). It should be of interest, in addition, to other agencies concerned with counter-insurgency research and/or the development of war-gaming techniques.

SUMMARY AND CONCLUSIONS

This Memorandum presents deterministic forms of Lanchester's equations as a model of small-force guerrilla engagements. Three types of military activity that are particularly characteristic of the early stages of insurgency are identified and analyzed:

1. Skirmish, in which both sides use maneuver, and surprise is not a major factor
2. Ambush, a surprise attack, causing weapon efficiencies on both sides to undergo rapid and significant change during the early stages of conflict
3. Siege, the attack of a fixed-perimeter fortification

The effects of troop captures and morale and/or discipline (as indicated by battle-stress desertions) are included in each case. Numerical solutions for illustratively chosen parameters are obtained by digital computer.

In the skirmish situations examined, discipline and morale, as well as weapon efficiency and force size, are shown to have a critical effect on the outcome, duration, and casualty production of the battle. In addition, it is shown that timely introduction of supporting weapons, particularly when the targets are densely clustered, can be of decisive influence. Although these effects are well known, the present treatment is novel in that the discipline/morale and supporting-weapon factors are treated explicitly, thereby enabling greater precision in predictions.

In the ambush situations examined, it is shown that in the absence of supporting weapons, the ambusher is usually successful against forces numerically 50 percent, and often 100 percent, larger. Even when the ambushee employs aggressive responsive tactics and ultimately causes the ambusher to break contact, the engagement can be a success for the ambusher, provided he breaks contact before the larger force can take proper advantage of its numbers. Despite the defender's use of rapid-response, short-duration supporting weapons, ambushes are still generally successful if they are well prepared and executed. A major deficiency of the rapid-response devices previously investigated is that

they concentrate on casualty production and not fire suppression. One possible technique for alleviating this deficiency is to substitute CS grenades for fragmentation grenades as the defensive mechanism.

Two stages of siege are postulated and analyzed: a softening-up barrage and an assault. The objectives of preliminary barrage are reduction of defensive artillery and softening of the defensive perimeter; the advantages of a preliminary barrage must be weighed against the loss of surprise which accompanies this action. The attacker's decision as to the advisability of pressing the follow-up assault is based on his estimate of the degree of neutralization of defensive artillery achieved, whether or not the defensive perimeter is negotiable, and whether or not the attack can be completed before defensive aircraft and heavy-artillery support can arrive. Descriptive equations are developed which can facilitate the attacker's decision-making procedure by enabling better estimates.

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SYMBOLS

- A = area over which targets are dispersed
- A_L = lethal area of supporting weapons
- A_T = presented area of personnel targets
- a, b, c = constants associated with troop discipline
- C = indifference parameter for skirmishes
- $E_{i,j}$ = supporting-weapon efficiencies (i-types on side n, j-types on side m)
- F = fraction of defending force remaining in Phase I of a siege
- f = constant associated with rate of casualties produced by supporting weapons
- G = offensive attrition parameter in Phase I of a siege
- H = unit step function
- K = defensive attrition parameter in Phase I of a siege
- k = direct-fire-weapon efficiencies
- \bar{k} = combined rate of fire of all supporting weapons of a single type
- k', k'' = constants associated with the shift from area fire to aimed fire
- m = numerical strength of infantrymen on side m (usually the defender or ambushee)
- n = numerical strength of infantrymen on side n (usually the aggressor or ambusher)
- $P_{H,K}$ = single-hit disablement probability
- P_K = kill or disablement probability of weaponry
- R = improvement in aiming capability achieved in time τ
- r = average rate of rifle fire
- S = killing power of ground forces engaged in battle
- S_c = time rate of casualties produced by supporting weapons of a single type

- T_s = time to fire \bar{k} rounds of supporting weapons of a single type
- t = time
- t_c = time required for discipline of an ambusher to deteriorate to the point where he may desert
- t_f = average time for a defender to fire after acquiring a target
- $W_{i,j}$ = supporting-weapon strengths (types i or j)
- α, β = constants associated with the time-dependency of a cover function
- γ = constant associated with the speed of shifting from area fire to aimed fire
- Δ = time delay associated with introduction of defensive artillery
- δ = constant reflecting severity of artillery attack on opposing artillery
- λ_n = constant associated with time for assaulter to acquire target
- ρ_l = areal density of fragments of mass group l
- σ = single-shot radial dispersion of fire
- τ = time required to achieve R improvement in aiming capability

Subscripts

- m = Side m (for ambush, ambushee; for siege, defender)
- n = Side n (for ambush, ambusher; for siege, attacker)

I. INTRODUCTION

In this Memorandum, deterministic forms of the Lanchester equations applicable to small-force guerrilla engagements are developed. The Lanchester equations⁽¹⁾ are based on the assumption that the attrition suffered by either side in conflict is a function of the strengths of the forces involved and the efficiency of their weapons. The equations are deterministic in that they are applicable, on the average, to a large number of similar engagements. Lanchester developed two cases: a linear law, purporting to represent ancient combat where individuals pair off and duel, and a square law which is more representative of the extended firepower capabilities of modern weapons. There is now an extensive body of literature devoted to Lanchester theory.⁽²⁻⁵⁾

Lanchester-type models especially applicable to certain forms of guerilla warfare were first proposed by Deitchman.⁽⁵⁾ A generalized model of such engagements, which includes the effects of time-dependent weapon efficiencies and which reflects battlefield desertions, captures, and supporting weaponry, is hypothesized herein. The general model is then specialized for three cases: skirmish, ambush, and siege. Typically, these categories account for most of the ground conflict in Phase II insurgency which has not yet escalated into traditional positional war, Phase III (see Mao Tse-tung, Ref. 6).*

It is recognized and emphasized that military-conflict models do not (and possibly cannot) properly reflect the full spectrum of terrorist, political, and sociological factors which can be associated with insurgency. They are therefore not sufficient for making predictions about the overall outcome (or even the Phase II outcome) of such conflicts. Military models are useful for lesser tasks, however; for example, with suitable models it appears possible to develop insights

*The first two phases of insurgency are characterized by small-force ground-yielding operations by the insurgents but overall military superiority on the part of the counterinsurgents. In Phase II the insurgent operations become increasingly military; however, they continue to be basically small-force guerrilla activities which cause the defense to fragment and the engagements to be localized and relatively isolated. In Phase III the insurgents take the strategic offensive and operate with larger, more conventional forces.

into the credibility of casualty estimates on both sides. It is clear that both the true casualty levels and the estimates of them (often used for propaganda purposes) are related to the outcome of the insurgency in important ways. Models of guerrilla engagements can also be useful when evaluating new military hardware, and they are of considerable academic interest as they relate to the historical development of war-gaming.

II. A MODEL OF THE MILITARY ASPECTS OF PHASE II INSURGENCY

In a Phase II insurgency, the military force of either side is hypothesized to consist of a large manpower pool from which small fighting groups are constantly being drawn for guerrilla-type operations. The manpower pool is organizationally structured but has substantial flexibility (particularly on the side of the insurgents) and, ideally, permits each fighting group to be brought to a desirable strength before engaging in an offensive operation.

The respective manpower pools consist of both voluntary belligerents and impressed neutrals. The neutrals remain in their pool until circumstances permit desertion. Under the stress of battle, desertions also occur both by individuals and by units.

Operations consist of a large number of small--typically 100-man--engagements that fall into three general categories: skirmishes, ambushes, and sieges. It is assumed that each operation is conducted in isolation, and for simplicity no ground-troop reinforcements are permitted during the course of an engagement. Conceptually, those engagements that are reinforced, and those engagements that include more than one operational category in the course of a single battle, are treated individually in the order in which they occur.

The flow of manpower is illustrated in Fig. 1. People engaged in military operations must, of course, be equipped with weapons, food,

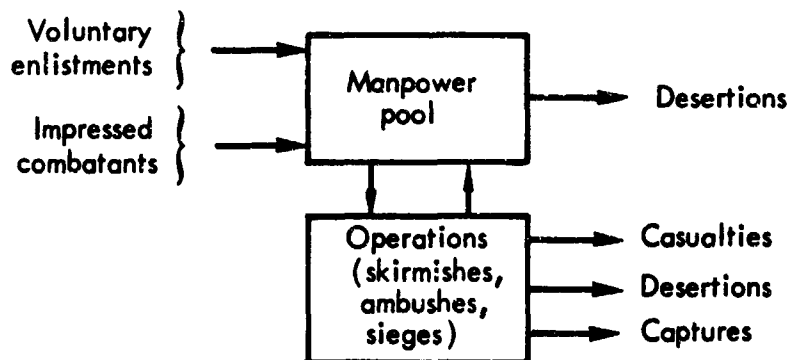


Fig.1—Military manpower flow in Phase II insurgency

and other necessities. In principle, a model and an accompanying flow chart can be developed for each militarily required item. For simplicity, we shall deal only with manpower. In effect, the food, weapon, and ammunition reserve of either manpower pool is assumed inexhaustible (this is largely justified by experience in Vietnam during 1963 and 1964).

The numerical strength of a manpower pool is therefore taken as the prime measure of the military strength of that side. As shown in Fig. 1, the manpower-pool strength is determined by direct recruitments and desertions, as well as by attrition due to operations. Traditional Lanchester theory deals only with that part of the operational attrition associated with casualties. In the present treatment, the traditional approach is modified by including operational desertions and captures, as well. No attempt is made to treat direct recruitments to and desertions from the manpower pools.

III. GENERALIZED LANCHESTER THEORY

Force depletion occurs in the small engagements in three ways: through casualties (including those killed in action and nonwalking wounded), surrender, and desertion. The total force-depletion rate is the sum of the rates from each of these sources.

Both sides are permitted supporting weapons. However, the insurgent supporting weapons (mortars and recoilless rifles) are light and highly portable, while counterinsurgent supporting weapons include artillery and ground-attack aircraft. No supporting-weapon duels are allowed (except in the preliminary stages of a siege). It is hypothesized that during Phase II, insurgents generally disengage rather than participate in this type of activity.

Assuming that a Lanchester-type law^{*} holds, on the average, for casualties, the attrition equations become

$$\left. \begin{aligned} \left(\frac{dm}{dt}\right)_c &= -k_n(t,m)n - \sum_i^1 E_i(t,m)W_i(t) \\ \left(\frac{dn}{dt}\right)_c &= -k_m(t,n)m - \sum_j^J E_j(t,n)W_j(t) \end{aligned} \right\} \quad (1)$$

where m and n are the numbers of engaged personnel on opposing sides, W_i and W_j are the supporting-weapon strengths, and t is a time-like

^{*}Lanchester's original equations can be expressed as

$$\left. \begin{aligned} \frac{dm}{dt} &= -S_n(m,n) \\ \frac{dn}{dt} &= -S_m(m,n) \end{aligned} \right\}$$

Upon integration, the ancient-combat version, $S_n = k_nmn$, $S_m = k_mmn$, leads to the linear law, whereas the modern-combat version, $S_n = k_nn$, $S_m = k_mm$, leads to the square law.

variable. It should be emphasized that the weapon-efficiency coefficients, k_n , k_m , E_1 , and E_j , are explicit functions of time and of the force level of the opposing side. They are all positive quantities such that $(dm/dt)_c$ and $(dn/dt)_c$ are less than zero. In general, W_1 and W_j are also functions of time, since supporting weapons are usually employed for only portions of the battle.

Two assumptions are made concerning the surrender and desertion rates:

1. Expected values of the rates can be predicted by similar deterministic laws. These two variables will, therefore, be combined into a single force-depletion term with the subscript $s+d$.
2. The rate of friendly surrenders and desertions depends on both the friendly casualty rate and the difference between the friendly force ratio and unity.

It is postulated that the surrender and desertion rates can be expressed as the sums of separate power series in these variables:

$$\left. \begin{aligned} \left(\frac{dm}{dt}\right)_{s+d} &= a_m - \left[b_{m1} \left(\frac{dm}{dt}\right)_c + b_{m2} \left(\frac{dm}{dt}\right)_c^2 + \dots \right] - \left[c_{m1} \left(\frac{n}{m} - 1\right) + c_{m2} \left(\frac{n}{m} - 1\right)^2 + \dots \right] \\ \left(\frac{dn}{dt}\right)_{s+d} &= a_n - \left[b_{n1} \left(\frac{dn}{dt}\right)_c + b_{n2} \left(\frac{dn}{dt}\right)_c^2 + \dots \right] - \left[c_{n1} \left(\frac{m}{n} - 1\right) + c_{n2} \left(\frac{m}{n} - 1\right)^2 + \dots \right] \end{aligned} \right\} \quad (2)$$

where

$$\left(\frac{dm}{dt}\right)_{s+d}, \left(\frac{dn}{dt}\right)_{s+d} \leq 0$$

$$a_m, a_n \leq 0$$

$$c_{m_i} = 0 \text{ for } m/n > 1 \quad (i = 1, 2, \dots)$$

$$c_{n_i} = 0 \text{ for } n/m > 1 \quad (i = 1, 2, \dots)$$

Since Eqs. (2) are to be added to Eqs. (1), a_m and a_n must be equal to or less than zero. (Otherwise, these terms could prevent casualties due to firepower from appearing in the model.) In a self-policing military group, it can be assumed that $a = 0$. Although this permits either or both surrenders and desertions as soon as a superior enemy force is encountered and/or as soon as casualties are sustained, it does not permit nonbattlefield defections or desertions from the fighting group.*

The coefficients b_m , b_n , c_m , and c_n reflect the training and motivation (discipline and morale) of the troops. Since the terms $(dm/dt)_{s+d}$ and $(dn/dt)_{s+d}$ are always negative or zero, the net sum of either the $b_{m,n}$ terms or $c_{m,n}$ terms must always be positive or zero (although conceptually, the individual coefficients can assume any real value). The greater the sum of either of these sets of terms, the poorer the motivation and discipline of the people involved.

As an approximation for computational purposes, only the first-order terms in $(dm/dt)_c$ and $(dn/dt)_c$ and the first- and second-order terms in $(n/m - 1)$ and $(m/n - 1)$ will be retained. The resulting generalized attrition equations are

$$\begin{aligned}\frac{dm}{dt} &= -(1 - b_m)k_n(t, m)n - c_{m_1}\left(\frac{n}{m} - 1\right) - c_{m_2}\left(\frac{n}{m} - 1\right)^2 - (1 - b_m)\sum_{i=1}^i E_i(t, m)W_i(t) \\ \frac{dn}{dt} &= -(1 - b_n)k_m(t, n)m - c_{n_1}\left(\frac{m}{n} - 1\right) - c_{n_2}\left(\frac{m}{n} - 1\right)^2 - (1 - b_n)\sum_{j=1}^j E_j(t, n)W_j(t)\end{aligned}\quad (3)$$

where

$$\frac{dm}{dt}, \frac{dn}{dt} \leq 0$$

$$b_m, b_n \leq 0$$

*As previously noted, defections or desertions can also occur directly from the manpower pool at any force level (see Fig. 1).

$$c_m = 0 \text{ for } m/n > 1$$

$$c_n = 0 \text{ for } n/m > 1$$

Note that when they incorporate properly chosen $c_{m,n}$ coefficients, the $(n/m - 1)$ and $(m/n - 1)$ terms can also simulate the act of breaking off an engagement. This is in keeping with the guerrilla tactic of fading into the jungle; i.e., when guerrilla forces are outnumbered or at some other disadvantage, they will gradually disengage, with the remaining troops fighting a rear-guard action. When the $c_{m,n}$ coefficients are used in this fashion, they also become arbitrary functions of time, since the action is deliberate and under the control of the unit commander.

Equations (3) are sufficiently general to represent a wide variety of guerrilla-engagement situations. Three special cases--skirmish, ambush, and siege--will be considered in the following sections.

IV. SKIRMISH

A skirmish, as defined herein, involves a relatively limited commitment of resources where the riflemen-engagement parameters are (on the average) independent of time. It is assumed that the riflemen of both sides engage in aimed fire. Supporting weapons, on the other hand, are employed to the extent that availability and tactics permit and thus continue to be explicit functions of time. The imposition of these requirements results in

$$\begin{aligned}\frac{dm}{dt} &= -(1 - b_m)k_n n - c_{m1}\left(\frac{n}{m} - 1\right) - c_{m2}\left(\frac{n}{m} - 1\right)^2 - (1 - b_m) \sum_1^1 E_1(t, m) W_1(t) \\ \frac{dn}{dt} &= -(1 - b_n)k_m m - c_{n1}\left(\frac{m}{n} - 1\right) - c_{n2}\left(\frac{m}{n} - 1\right)^2 - (1 - b_n) \sum_j^j E_j(t, n) W_j(t)\end{aligned}\quad (4)$$

No general solution of the skirmish equations is available. However, accurate numerical solutions can be obtained easily with high-speed digital computers. The outcomes of an illustrative set of such numerical solutions are given in Tables 1 through 3. Table 1 (p. 12) covers a wide set of cases where there are no support weapons (i.e., $W_1 = W_j = 0$), while Tables 2 and 3 (pp. 20 and 22) illustrate a few cases where support weapons are introduced on one side. In addition to allowing the initial force ratios to vary significantly, these solutions also include the effects of variations in the weapon-efficiency and discipline/morale coefficients $(1 - b_m)k_n$, $(1 - b_n)k_m$, and c_m, c_n .^{*}

^{*}In these and subsequent solutions, the coefficients c_{m1} and c_{n1} have been arbitrarily set at zero. The omission of one of the terms is done strictly for convenience, since retention of both hinders orderly presentation of the results. Note that omission of the linear term makes desertions relatively insensitive to n/m and m/n when these ratios are close to unity, and highly sensitive to them when they are large compared to unity. Such a formulation can more realistically simulate the breaking off of an engagement than can the use of the linear term alone, and therefore we have retained the second-order term. However, the broad conclusions would be the same if the reverse had been done. In the remainder of this Memorandum, c_m and c_n refer to c_{m2} and c_{n2} .

Tables 1 through 3 show typical values for the weapon-efficiency and discipline/morale coefficients. For purposes of illustration, this assignment of parameters can be interpreted as follows.*

Weapon Efficiency: $(1 - b_{m,n})k_n, (1 - b_{n,m})k_m = 0.04, 0.06, 0.08$

Assume the $k_{m,n}$ are fixed at 0.04. This would require $b_{m,n}$ to assume the values 0, -0.5, and -1, and it would imply that as the result of every casualty, $-b_{m,n}$ troops surrender or desert. Although the values of $k_{m,n}$ are assigned arbitrarily, they are reasonable, as shown below.

From elementary probability considerations, the weapon-efficiency coefficients for aimed fire can be approximated by

$$\left. \begin{aligned} k_n &= r_n P_{K_n} = \frac{r_n A_{T_m} P_{H,K}}{2\pi\sigma_n^2} \\ k_m &= r_m P_{K_m} = \frac{r_m A_{T_n} P_{H,K}}{2\pi\sigma_m^2} \end{aligned} \right\} \quad (5)$$

where

- r_n, r_m = average rates of rifle fire; an illustrative value of 5 rds/min would permit expenditure of 10 lb of .22-cal rifle ammunition in about 80 min
- A_{T_m}, A_{T_n} = presented area of prone infantryman to rifle fire over average terrain; typically, 0.1 ft^2 at a range of 100 ft
- $P_{H,K}$ = single-hit disablement probability to rifle fire; a typical value is 0.5

*The reader is cautioned against accepting the illustrative interpretation literally, since the parameters have been oversimplified for brevity. For example, the presented area of infantrymen (A_T), the single-hit disablement probability ($P_{H,K}$), and single-shot radial dispersions of fire (σ) are actually functions of range and of whether an attack or defense posture is being maintained.

σ_n, σ_m = single-shot radial dispersions of fire; a value of 1 ft corresponding to 10 mils at 100-ft range can be considered typical

Discipline/Morale: $c_m, c_n = 0, 1$

These coefficients reflect desertions associated with being outnumbered. For example, the value of 1 indicates that when outnumbered 2 to 1, the weaker side will, on the average, desert at the rate of one per minute because of individual assessments of the local force ratios.

* * * * *

When the initial force disparity is 1.5 to 1 or more (see case $m_o = 75$ in Table 1, where the subscript o represents the initial force levels), the major effect of desertions or surrenders is to change the duration but not the outcome of the conflict. For the 36 parametric combinations investigated, the numerically superior Side m emerges the victor in each case. As would be expected, the time to achieve these victories decreases as $(1 - b_n)k_m$ increases; this implies simply that as the Side m weapon efficiency increases and/or as the Side n casualty-related discipline/morale decreases, the victories are more easily achieved for Side m. Conversely, as $(1 - b_m)k_n$ increases, the time to achieve the victories increases.

Within the range of $(1 - b_{m,n})k_{n,m}$ investigated for the initial force ratio of 1.5 to 1, Side n never gains the force advantage. It follows then that for these cases, differences in the coefficients c_n and c_m (i.e., desertions due to being outnumbered) can never affect the final outcome, since they operate in favor of the initially weak force only when it gains the force (manpower) advantage. Note, however, that the time associated with the conflict decreases as c_n increases. This indicates that as the Side n rate of desertion due to being outnumbered increases, the victories become easier for Side m.

The most obvious way to increase the probability of victory for Side n is to decrease the initial force ratio m_o/n_o . To achieve such variations Side n can deliberately choose to avoid particular engagements if it is outnumbered beyond a certain ratio. From Table 1, for $m_o = 65$ ($m_o/n_o = 1.3$) it can be seen that Side m continues to win if

Table 1
SAMPLE SKIRMISH RESULTS WHERE NO SUPPORTING WEAPONS ARE USED
($n_0 = 50$)

Coefficient		$c_m = c_n = 0$		$c_m = 0, c_n = 1$		$c_m = 1, c_n = 0$		$c_m = 1, c_n = 1$	
$(1 - b_p)k_n$	$(1 - b_n)k_m$	Winner	Time (min)	Winner	Time (min)	Winner	Time (min)	Winner	Time (min)
$n_0 = 75$									
0.04	0.04	■	19.4	■	8.5	■	19.4	■	8.5
0.04	0.06	■	12.0	■	6.0	■	12.0	■	6.0
0.04	0.08	■	8.7	■	4.7	■	8.7	■	4.7
0.06	0.04	■	22.6	■	9.7	■	22.6	■	9.7
0.06	0.06	■	12.9	■	6.4	■	12.9	■	6.4
0.06	0.08	■	9.2	■	4.9	■	9.2	■	4.9
0.08	0.04	■	30.0	■	12.0	■	30.0	■	12.0
0.08	0.06	■	14.2	■	7.1	■	14.2	■	7.1
0.08	0.08	■	9.7	■	5.2	■	9.7	■	5.2
$n_0 = 65$									
0.04	0.04	■	24.5	■	12.2	■	24.5	■	12.2
0.04	0.06	■	14.5	■	8.0	■	14.5	■	8.0
0.04	0.08	■	10.4	■	6.1	■	10.4	■	6.1
0.06	0.04	■	34.6	■	16.9	■	34.6	■	16.9
0.06	0.06	■	16.4	■	9.1	■	16.4	■	9.1
0.06	0.08	■	11.2	■	6.6	■	11.2	■	6.6
0.08	0.04	■	27.0	■	27.9	■	21.2	■	22.0
0.08	0.06	■	19.7	■	10.9	■	19.7	■	10.9
0.08	0.08	■	12.3	■	7.3	■	12.3	■	7.3
$n_0 = 50$									
0.04	0.04	Draw	■	Draw	■	Draw	■	Draw	■
0.04	0.06	■	22.5	■	14.8	■	22.5	■	14.8
0.04	0.08	■	15.0	■	10.1	■	15.0	■	10.1
0.06	0.04	■	22.5	■	22.5	■	14.8	■	14.8
0.06	0.06	Draw	■	Draw	■	Draw	■	Draw	■
0.06	0.08	■	18.3	■	12.6	■	18.3	■	12.6
0.08	0.04	■	15.0	■	15.0	■	10.1	■	10.1
0.08	0.06	■	18.3	■	18.3	■	12.6	■	12.6
0.08	0.08	Draw	■	Draw	■	Draw	■	Draw	■

$(1 - b_m)k_n = 0.04$ or 0.06 , although it does so with increasing difficulty; e.g., the times to achieve the victories are longer. Finally, for $(1 - b_m)k_n = 0.08$, $(1 - b_n)k_m = 0.04$, Side n commences to be the victor.

It is apparent from these considerations that for every m_0/n_0 there must exist a combination of coefficients $(1 - b_m)k_n$ and $(1 - b_n)k_m$ for which either side can win; i.e., an indifference point. Furthermore, this indifference condition will depend on the value of c_n . In general, for $m_0 > n_0$, the larger the value of c_n , the larger the $(1 - b_m)k_n$ required to develop an indifference point. When $m_0 = n_0$ (see Case $m_0 = 50$ in Table 1), c_n has no influence on the indifference requirements; they are determined completely by equivalence of the coefficients $(1 - b_m)k_n$ and $(1 - b_n)k_m$.

It is of interest to generalize the foregoing discussion for the skirmish case with no support weapons. Taking $m_0 > n_0$ and considering only the period for which $m > n$, we have

$$\left. \begin{aligned} \frac{dm}{dt} &= -(1 - b_m)k_n n \\ \frac{dn}{dt} &= -(1 - b_n)k_m m - c_n \left(\frac{m}{n} - 1\right)^2 \end{aligned} \right\} \quad (4a)$$

$$\Delta m = \frac{(1 - b_m)k_n n \Delta n}{(1 - b_n)k_m m + c_n \left(\frac{m}{n} - 1\right)^2} \quad (4b)$$

Now the criterion for a Side n victory under the above circumstances must be that $(m - \Delta m)/(n - \Delta n) < m/n$ for all m, n . Combining this inequality with Eq. (4b) and rearranging terms, the criterion for a Side n victory when $m_0 > n_0$ becomes

$$1 - \frac{c_n \left(\frac{m}{n} - 1\right)^2}{(1 - b_m) k_n n^2} > \frac{(1 - b_n) k_m}{(1 - b_m) k_n} \frac{m^2}{n^2} \quad (6)$$

for all values of m, n . *

Thus, the numerically weaker side will win whenever the discipline/morale factors irreversibly outweigh the firepower disparities. However, satisfying Inequality (6) at the initial conditions does not guarantee that the inequality holds throughout the battle. It is, in fact, necessary to integrate the skirmish Eqs. (4a), varying the coefficients in a systematic manner, if one wishes to determine the indifference points. It turns out that the dimensionless ratios m_o/n_o , $k_m(1 - b_n)/k_n(1 - b_m)$, and $c_n/k_n(1 - b_m)n_o$, suggested by Inequality (6), can be used to correlate the results (see Fig. 2).

The indifference curves of Fig. 2 represent the particular combinations of parameters for which either side can win. Note that all space above the $C = 0$ curve represents Side m victories; this is the square-law result corresponding to $c_n = 0$ (see footnote below), where the outcome is determined by Inequality (6a). For $C \geq 0$, corresponding to any m_o/n_o , one can always find a $k_m(1 - b_n)/k_n(1 - b_m)$ that will produce an indifference point. For example, when $m_o/n_o = 1.3$ (see Case $m_o = 65$ in Table 1), Side n victories were produced when $k_m(1 - b_n)/k_n(1 - b_m) = 0.04/0.08 = 0.5$; however, only the range of c_n from 0 to 1 was evaluated. The indifference point would thus require a value of $c_n > 1$. This conclusion is confirmed in Fig. 2, which shows that for the coordinate value (1.3, 0.5), indifference is achieved with $C = 1$, which corresponds to a value of $c_n = 4.0$. Note also that

* Note that when $c_n = 0$, Inequality (6) immediately gives the square-law result: (2,3)

$$\frac{(1 - b_m) k_n}{(1 - b_n) k_m} > \frac{m_o^2}{n_o^2} \quad (6a)$$

for a Side n victory. In this special case, the initial force ratio determines the outcome, since the ratio (m/n) varies monotonically with time.

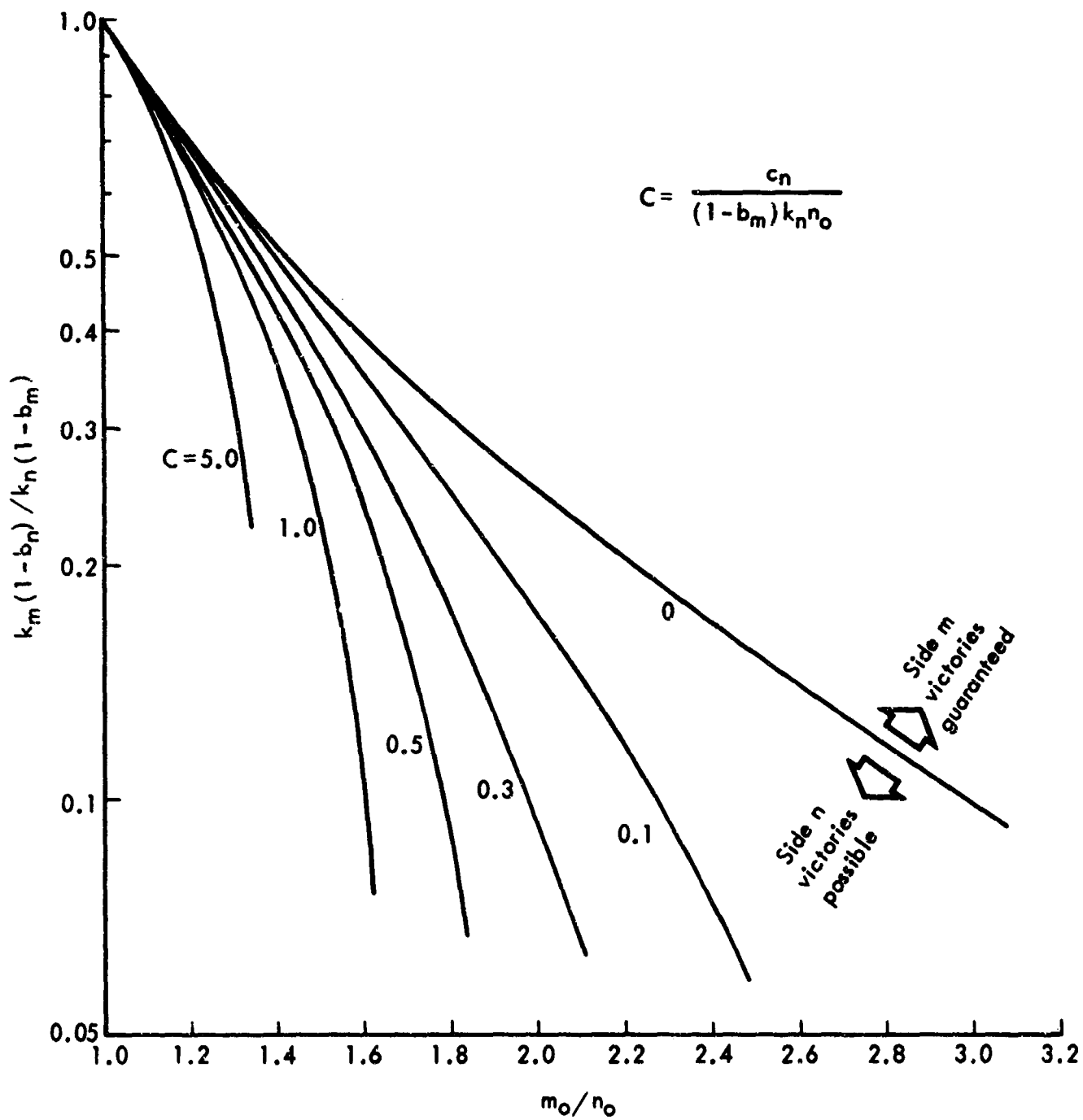


Fig.2—Indifference curves for skirmish (no support weapons)

for $m_0 = n_0$, all indifference curves pass through the point (1.0, 1.0) irrespective of the value of c_n .

Thus far, the discussion has been essentially confined to determining the winner of the conflict. No mention has been made of the casualties incurred by either side, or of the battlefield desertions or surrenders. Unfortunately, few generalities can be made about these topics because they are not time-independent (as are the indifference curves of Fig. 2). We will therefore develop some insight into such force depletion by means of a few examples.

It was shown earlier that when the initial force disparity is approximately 1.5 to 1 or more, the major effect of desertions or surrenders is to change the duration of battle but not the outcome. A secondary effect in these cases is that the casualties on the stronger side either increase or decrease, depending on whether the conflict is lengthened or shortened. By way of contrast, with only minor exceptions the casualties on the weaker side either decrease or remain the same. (Insurgents take advantage of this tendency by gradually disengaging to conserve their forces.)

The foregoing effects are illustrated in Fig. 3, where representative solutions are plotted, covering the range of morale/discipline factors investigated. For interpretive purposes, we will assume that both sides have equivalent weapon effectiveness ($k_{m,n} = 0.04$). Obviously, the skirmish in which the larger (m) force has excellent discipline and the smaller force poor discipline will terminate first. (Case 1 corresponds to $b_m = 0$, $b_n = -1$, $c_n = 1$.) When the $b_{m,n}$ morale/discipline factors are nonzero and equal (Cases 2 and 3), the contests are again shortened, although to a lesser extent. In all cases where the engagement is shortened, there are fewer Side m casualties than in those cases governed by square-law conflict ($b_{m,n} = 0$, $c_{m,n} = 0$).

When the b_m term substantially dominates the b_n term (the smaller force is better disciplined than the larger force), the contest is prolonged--and correspondingly, the larger force suffers greater casualties (Cases 5 and 6). Note that the small-force discipline must be near-perfect ($b_n, c_n \sim 0$) for this condition to hold. In Case 4, for example, where $b_m = -1$ and $b_n = -0.5$, the contest is shortened.

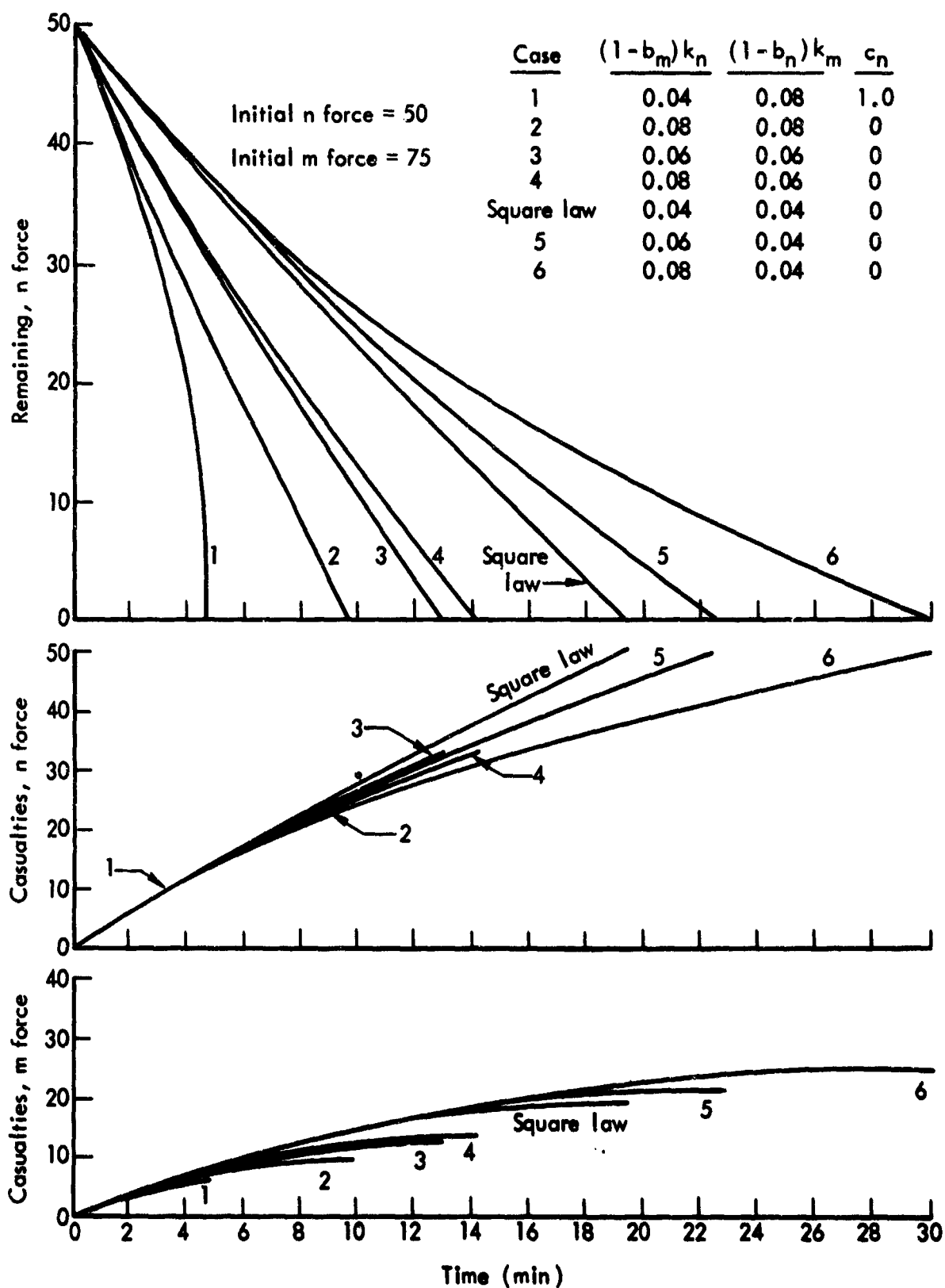


Fig.3—Skirmish illustrations (no support weapons)

The qualitative features demonstrated for $m_0/n_0 = 1.5$ are quite general and apply to a wide range of cases where $m_0 > n_0$, although details can differ. The one restriction is that Side m must continue to outnumber Side n and must eventually win the battle. When Side m does not emerge the winner (see Fig. 2), the remarks apply only up to the time of numerical equality, after which the opposite holds true. The statements regarding the initially stronger force are subsequently applied to the initially weaker force (with due recognition given the concomitant requirement that $c_n = 0$ and $c_m \geq 0$). Unfortunately, the net results regarding casualties for the inverted cases cannot be generalized, and they must be considered individually for details.

The effect of supporting weapons of a single type backing up the weaker side is illustrated in Tables 2 and 3. For the initial condition of $m_0 > n_0$, letting $\sum_1 E_1(t, m) W_1(t) = S_c(t, m)$, we have

$$\left. \begin{aligned} \frac{dm}{dt} &= -\left(1 - b_m\right)\left[k_n n + S_c(t, m)\right] - c_m\left(\frac{n}{m} - 1\right)^2 \\ \frac{dn}{dt} &= -\left(1 - b_n\right)k_m(S_c)_m - c_n\left(\frac{m}{n} - 1\right)^2 \end{aligned} \right\} \quad (7)$$

Note that in Eq. (7), $k_m = k_m(S_c)$, which permits the well-known effect that supporting weapons not only can produce direct enemy casualties but can also reduce friendly casualties, desertions, and surrenders by suppressing enemy fire.

We assume for simplicity that the supporting weapons engage the enemy with area fire; i.e., they are not aimed at specific targets but randomly cover the area in which the enemy is dispersed. Assuming further that the Side m targets are randomly dispersed in area A_m , neglecting edge effects, and letting A_L be the lethal area of the supporting weapons and \bar{k} the combined rate of fire of the supporting weapons, we have

$$S_c \approx \left(\frac{\bar{k} A_L}{A_m} \right) m = (f)m \quad (8)^*$$

Corresponding to the 105-mm howitzer batteries represented in Tables 2 and 3, \bar{k} is 15 rds/min, and A_L is 600 ft²/rd. We further estimate the Side m force ($m_0 = 75$) to be dispersed over areas ranging from 50,000 to 500,000 ft². These values combine roughly to produce values for f varying from 0.2 to 0.02/min.

Table 2 also includes two time-dependent cases: $S_c = 0.02mH(t - 5)$ and $0.2mH(t - 5)$, where H is the unit step function

$$H = \begin{cases} 0, & t < 5 \\ 1, & t \geq 5 \end{cases}$$

Here we recognize the possibility of delay in bringing the support weapons into the battle. The delay of 5 min is, of course, illustrative.

In Table 2 it is shown that for a typical set of parameters, ($n_0 = 50$, $m_0 = 75$, and $c_n = c_m = 1$), the effect of force dispersal is critical when supporting weapons are introduced at the beginning of battle. When the Side m force is well dispersed (about 80 ft between men initially, $f = 0.02$), the introduction of a battery of artillery changes the outcome in only one case: $(1 - b_m)k_n = 0.08$, $(1 - b_n)k_m = 0.04$, corresponding to poor discipline for the large force and good discipline for the small force and/or better weapon efficiencies for the smaller force. (Compare this with the $c_m = 1$, $c_n = 1$ column in Table 1.) On

* Equation (8) holds approximately if $\bar{k} A_L \approx 0.2 A_m$. A more exact formula accounting for overlapping effects would be

$$S_c = \left[1 - \left(1 - \frac{A_L}{A_m} \right)^{E T_s} \right] \frac{m}{T_s} \quad (8a)$$

where T_s is the time it takes to fire \bar{k} rounds. A precise definition of A_L is given in Eq. (11).

the other hand, when the Side m force is poorly dispersed (about 25 ft between men, $f = 0.2$), the outcome is changed in every case. However, even in the cases where the outcome is not changed, the battle is prolonged, and consequently, casualties are increased in the larger force.

Table 2 also shows that a delay in introducing the supporting weapon^s even as short as 5 min, can be of critical importance in deciding the contest. Six of the ten cases where the outcome had been reversed now revert to the original result. Therefore, for the set of parameters considered, the commander of the large force can generally resort to concentration of forces, with the expectation of quick victory, if he knows that his adversary will be delayed in bringing his artillery to bear and his own forces are better or equally as disciplined as the enemy.

Finally, the suppressive-fire effects of supporting artillery are considered illustratively in Table 3; the same parameters explored in Table 2, given the 5-min delay, are employed. In addition, we introduce the following dependency in the large-force weapon efficiencies: $k_m = 0.04 - 0.01H(t - 5)$. In effect, it is assumed that when the supporting weapons actually fire, the accuracy of the large-force aimed fire is cut (roughly) in half, or alternatively, the large-force rate of fire is cut by 25 percent. Combinations of the two interpretations are also possible.

For the combination of parameters investigated, the suppressive-fire effects do not appear to be as important as the other variables considered. One outcome is reversed for the dispersed force (the case where $(1 - b_m)k_n = 0.08$ and $(1 - b_n)k_m = 0.04$ in Tables 2 and 3) and one for the concentrated force (the case where $(1 - b_{m,n})k_{n,m} = 0.06$); both of these cases had previously proved to be sensitive to supporting-weapon delay as well. In all other cases where the delayed introduction of artillery did not change the outcome, the suppressive fire produced no effect other than to lengthen the conflict (and thereby increase Side m casualties).

Table 3
SAMPLE SKIRMISH RESULTS WHERE SUPPORTING WEAPONS ARE USED (INCLUDING SUPPRESSIVE-FIRE EFFECTS)^a
($a_0 = 50$, $m_0 = 75$)

Coefficient		Combined Supporting-Weapon-Effectiveness Rate, ^b $f = S_c/m$			
		0.02H(t - 5)		0.2H(t - 5)	
$(1 - b_m)k_n$	$(1 - b_n)k_m$	Winner	Time (min)	Winner	Time (min)
0.04	0.04	m	9.6	n	13.8
0.04	0.06	m	6.1	m	6.5
0.04	0.08	m	4.7	m	4.7
0.06	0.04	m	13.1	n	10.4
0.06	0.06	m	6.7	n	13.2
0.06	0.08	m	4.9	m	4.9
0.08	0.04	n	20.8	n	8.7
0.08	0.06	m	7.7	n	9.8
0.08	0.08	m	5.2	m	5.3

^a Side n possesses the equivalent of three 105-mm howitzers; $k_m = 0.04 - 0.01H(t - 5)$;
 $c_n = c_m = 1$.
 $b_n = b_m = m(t)$.

To summarize this discussion of small-force skirmishes,* it has been demonstrated analytically that morale and discipline, as well as weapon efficiency and force disparity, can have a critical effect on the outcome of battle. It has also been shown that the duration of battle and, consequently, the number of casualties produced are significantly affected by these parameters. Finally, it has been demonstrated that timely introduction of supporting weapons can sway the course of battle. The level of dispersion of opposing forces appears to determine the degree of importance of the supporting weapons in a particular battle.

These results are, of course, well demonstrated by experience and military history. The novelty of the present treatment is that the morale/discipline and supporting-weapon factors are treated explicitly. In small-force guerrilla operations, where morale and discipline often dominate the battle scene, a commander can thus make better estimates of the outcome.

*It is noted in passing that, conceptually, skirmishes could occur under conditions where the infantrymen on one or both sides engage in area fire rather than the aimed fire assumed. Such conditions would change the relevant equations to variants of Lanchester's linear law. However, using procedures similar to those used in the foregoing discussion, it can be demonstrated that the broad conclusions do not change.

V. AMBUSH

In the ambush, in contrast to the skirmish, the time-dependence of the weapon-efficiency coefficients is important and perhaps dominant. This time-dependency results from the changing cover (shielding) available to the individuals on the defensive side and the defense's gradual transition from area to aimed fire when it responds to the attack.

Because of the surprise element in an ambush, defensive cover is initially minimal. As the engagement progresses, the ambushee seeks whatever cover is available and gradually improves his situation. The attackers, on the other hand, have a relatively secure position which remains constant until the contest ends (or until they choose to break off the engagement).

The ambushees generally enter the contest by engaging in area fire, because of their lack of preparation for the immediate conflict. However, as the battle unfolds, the defense maneuvers, attempts to locate the attackers, rushes the opponent's position if possible, and gradually switches from area to aimed fire.* The ambushers, on the other hand, engage in aimed fire throughout, although its net quality deteriorates with time.

In the early stages of the ambush, there is little motivation for those on the attacking side to desert or surrender. It is reasonable to suppose, however, that as the contest progresses the discipline of the ambushers deteriorates somewhat. Alternatively, it is possible that the ambusher might make a deliberate decision to commence a gradual withdrawal after a specific period of time. These variations from the general situation described earlier can be handled by introducing the coefficient $c_n(t)$ defined as

*This description of the early stages of an ambush is the same as that presented by Deitchman.⁽⁵⁾ Deitchman describes a static situation where the ambushees engage in area fire and the ambushers in aimed fire. In the Lanchester sense, this is a mixed linear-square law. The present concept is initially a mixed linear-square law which gradually changes to a pure square law.

$$c_n(t) = |c_n| H(t - t_c) H\left(\frac{m}{n} - 1\right)$$

i.e., $c_n(t)$ is a positive quantity when $t > t_c$ and $m/n > 1$, and it is zero otherwise. The ambush equations then become

Ambushee Attrition

$$\frac{dm}{dt} = -(1 - b_m) k_n(t) n - c_m \left(\frac{n}{m} - 1\right)^2 - (1 - b_m) \sum_1^j E_i(t, m) W_i(t)$$

Ambusher Attrition

(9)

$$\frac{dn}{dt} = -k_m(n, t) m - c_n(t) \left(\frac{m}{n} - 1\right)^2 - \sum_j^j E_j(t, n) W_j(t)$$

By analogy to Eqs. (5), the ambusher small-arms weapon-efficiency coefficients become

$$k_n(t) = \frac{r_n A_T(t) P_{H,K}}{2\pi\sigma_n^2} \quad (5a)$$

with a reasonable representation of $A_T(t)$ being

$$A_T(t) = \frac{A_{T\infty}}{1 - e^{-\alpha t - \beta}} \quad (10)$$

In Eq. (10), $A_{T\infty}$ is the steady-state value of the defensive-cover function (minimum presented area), and α and β reflect the speed with which the ambushee can approach the level of this maximum cover. A typical value for $A_{T\infty}$ for prone troops against rifle fire is 0.1 ft^2 .

By analogy to Eq. (8), the casualties produced by the supporting weapons of the ambusher are given by

$$\sum_i E_i(t, m) W_i(t) \approx \frac{1}{A_m} \sum_i \bar{k}_i(t) \int_A m P_K(t) dA \quad (11)$$

$$\approx \frac{\bar{k}(t) A_L(t) m}{A_m} \quad (8a)$$

if the weapons are of a single type and the Side m force is randomly distributed. In Eq. (11), A is the area of effect of a single round and P_K is the kill probability in an increment of area.

In general, the dependence of P_K on time arises through $A_T(t)$. For example, if fragmentation weapons are involved,

$$P_K(t) = 1 - \exp \left[-A_T(t) \sum_l \rho_l P_{HK_l} \right] \quad (12)$$

where ρ is the areal density of fragments of mass group l , P_{HK_l} is the single-hit kill probability of these fragments, and $A_T(t)$ is of the form of Eq. (10). When Eq. (10) is interpreted for prone troops against high-explosive fragmentation weapons, a typical value for the average presented area of targets under steady-state conditions ($A_{T\infty}$) is 0.5 ft^2 .

Turning to the ambushee weapon coefficients, $\sum_j E_j(t, n) W_j(t)$ is of the same form as Eq. (11) except that P_K is not an explicit function of time. However, the small-arms weapon coefficient $k_m = k_m(n, t)$ is explicitly time-dependent, since there is a gradual transition from area to aimed fire.

A reasonable representation of $k_m(n, t)$ which simulates this transition is

$$k_m(n, t) = k''(1 - e^{-\gamma t}) + k'n e^{-\gamma t} \quad (13)$$

Note that when $t = 0$, we have $k_m = k'n$ (the appropriate form for area fire), and when $t = \infty$, $k_m = k''$ (the appropriate form for aimed fire).

The constant γ can be interpreted as follows: As a result of better aiming, it is desired to improve k_m by a factor of R in time τ ; i.e., since aiming is reflected in k' or k'' by the reciprocal of σ^2 ,

$$\left[\frac{\sigma(0)}{\sigma(\tau)} \right]^2 = R \quad (14)$$

whence

$$\frac{k''(1 - e^{-\gamma\tau}) + k'n_0e^{-\gamma\tau}}{k'n_0} = R \quad (15)$$

and

$$\gamma = \frac{1}{\tau} \ln \left[\frac{1 - \frac{nk'}{k''}}{1 - \frac{n_0 k' R}{k''}} \right] \quad (16)$$

In practice,

$$\frac{nk'}{k''} < \frac{n_0 k'}{k''} \ll 1$$

Hence

$$\gamma \approx \frac{1}{\tau} \ln \left[\frac{1}{1 - \frac{n_0 k' R}{k''}} \right] \quad (17)$$

Representative numerical solutions of the ambush equations have been carried out, again by digital computer. For purposes of illustration, values for α and β of 6.2/min and 0.1, respectively, have been assumed; this implies that the targets achieve approximately 95 percent of their eventual cover within 0.5 min. For illustrative purposes, it is sufficient to assume typical values of the weapon

efficiencies (here, we have assumed $k_n(\infty) = 0.06$, corresponding to an 8-mil aiming error by the ambusher, and all the other conditions of the skirmish illustrations; similarly, $k_m(\infty) = k'$ has been fixed at 0.06). Again, for illustration, the ambushers are given the capability of maintaining perfect discipline for 10 min, i.e., $t_c = 10$.

The results for illustrative cases with no support weapons on either side are summarized in Tables 4a, 4b, and 4c (pp. 29, 32, and 33). A two-order-of-magnitude range of initial weapon-efficiency disparity has been covered; that is, $k_m(\infty)/k_m(o)n_o = 50$ to 5000. Throughout this range, the effect of the ambushee's speed in shifting from area to aimed fire has been investigated in detail, i.e., aiming recoveries* of 10, 20, and 40 percent and recovery times of 5, 10, and 15 min have been used as parameters. In addition, a wide range of discipline/morale factors on both sides has been examined.

As shown in Table 4a, where the ambushers (Side n) engage a force twice their size, it is clear that neither the outcome nor the duration of the contest is very sensitive to the initial weapon-efficiency disparity, provided the ambusher maintains good discipline. The factors that do appear to be of major importance are (1) the speed with which the ambushees (Side m) develop aiming capabilities (weapon efficiencies) approaching those of the attackers, and (2) the level of discipline maintained by the ambushees. For example, when Side m maintains perfect discipline ($b_m = 0$, $c_m = 0$), it emerges the victor, provided 40 percent of the enemy's individual weapon performance ($R = 20$ or 2000) is achieved in 15 min or less; Side m also wins if it achieves 20 percent of the enemy's efficiency ($R = 10$ or 1000) within 5 min. However, when $b_m = -0.5$ (one desertion or surrender for every two casualties), Side m must typically achieve 40 percent of the enemy's efficiency within 5 min in order to win. (Note that the value of c_m can affect the duration but not the outcome of these contests.)

Desertions and surrenders among the ambushers (as embodied in c_n) do not appear to be a major factor in determining the outcome, provided

* Aiming recovery = $\frac{R}{k_m(\infty)/k_m(o)n_o}$ = fraction of ultimate weapon effectiveness.

Table 4a

SAMPLE AMBUSH RESULTS WHERE NO SUPPORTING WEAPONS ARE USED^a
($n_o = 50, m_o = 100$)

τ (min)	$k_m(\omega)/k_m(o)n_o = 5000$						$k_m(\omega)/k_m(o)n_o = 50$					
	R = 2000			R = 1000			R = 500			R = 200		
	Winner	Time (min)		Winner	Time (min)		Winner	Time (min)		Winner	Time (min)	
$b_m = 0, c_m = 0, c_n = 0.5$												
5	m	13.8	m	n	21.4	n	n	47.8	m	m	13.5	m
10	m	19.6	n	n	51.3	n	n	36.5	m	n	18.9	n
15	m	26.9	n	n	39.5	n	n	34.6	m	n	25.1	n
$b_m = 0, c_m = 0, c_n = 1$												
5	m	12.6	m	n	18.4	n	n	50.7	m	m	12.4	m
10	m	17.1	n	n	57.7	n	n	36.7	m	m	16.5	m
15	m	22.5	n	n	40.0	n	n	34.7	m	m	21.1	n
$b_m = -0.5, c_m = 0, c_n = 0.5$												
5	m	18.0	n	n	27.9	n	n	23.4	m	n	17.5	n
10	n	29.8	n	n	23.6	n	n	22.1	n	n	31.3	n
15	n	25.5	n	n	22.7	n	n	21.8	n	n	26.2	n
$b_m = -0.5, c_m = 0, c_n = 1$												
5	m	16.0	n	n	27.9	n	n	23.5	m	n	15.6	n
10	n	30.0	n	n	23.6	n	n	22.2	n	n	31.5	n
15	n	25.5	n	n	22.7	n	n	21.8	n	n	26.2	n
$b_m = -0.5, c_m = 1, c_n = 0.5$												
5	m	18.0	n	n	23.3	n	n	19.4	m	n	17.5	n
10	n	25.1	n	n	19.6	n	n	18.3	n	n	26.4	n
15	n	21.3	n	n	18.7	n	n	18.0	n	n	21.9	n
$b_m = -0.5, c_m = 1, c_n = 1$												
5	m	16.0	n	n	23.4	n	n	19.5	m	n	15.6	n
10	n	25.3	n	n	19.6	n	n	18.3	n	n	26.6	n
15	n	21.3	n	n	18.7	n	n	18.0	n	n	21.9	n

^a $k_n(\omega) = k_m(\omega) = 0.06; \tau_c = 10.$

these can be postponed for about 10 min and provided the initial weapon-efficiency disparity is large. When $k_m(\infty)/k_m(o)n_o = 5000$, for example, the contests are either shortened (for Side m victories) or lengthened (for Side n victories), but not reversed, as c_n is varied from 0.5 to 1.0. On the other hand, the outcome of several contests changes when $k_m(\infty)/k_m(o)n_o = 50$; for example, the $R = 10$, $\tau = 10$ result for perfect discipline is changed from a Side n victory in 69.5 min to a Side m victory in 36.0 min. One would also expect the results to be sensitive to smaller values of t_c .

The foregoing effects are illustrated in Fig. 4, where remaining forces and Side m casualties are plotted against time over the range of important parameters, under the conditions of Table 4a. The square-law results are again included as a point of reference. Because of the many special conditions of Fig. 4 (and Table 4a), it is difficult to generalize other than to emphasize the tremendous advantages accruing to the ambusher (Side n). In many cases, these are sufficient to give him a victory, but even if he eventually loses or disengages, the ambusher extracts a heavy toll compared to that in square-law conflict.

As the initial force disparity decreases to about 1.5 to 1 (see the case of $n_o = 50$, $m_o = 75$ in Table 4b), the frequency of Side n victories increases significantly. At this force level, even with perfect discipline, Side m must achieve 40 percent of the enemy's efficiency within 5 min in order to win. The generalities observed in Table 4a also apply here, except that outcomes and durations are even less sensitive to initial weapon-efficiency disparities and to ambusher discipline/morale factors.

When initial forces are equal (see Table 4c), in the absence of supporting weapons and for the assumed conditions of steady-state weapon-efficiency parity, $k_n(\infty) = k_m(\infty)$, the ambusher must in theory always win. Of course, the duration of the contest remains a variable, although it is relatively insensitive to both the initial weapon-efficiency disparity and the speed of aiming recovery by the ambushee. The duration of the contest is sensitive to the level of discipline maintained by the ambushee: The poorer the ambushee discipline, the shorter

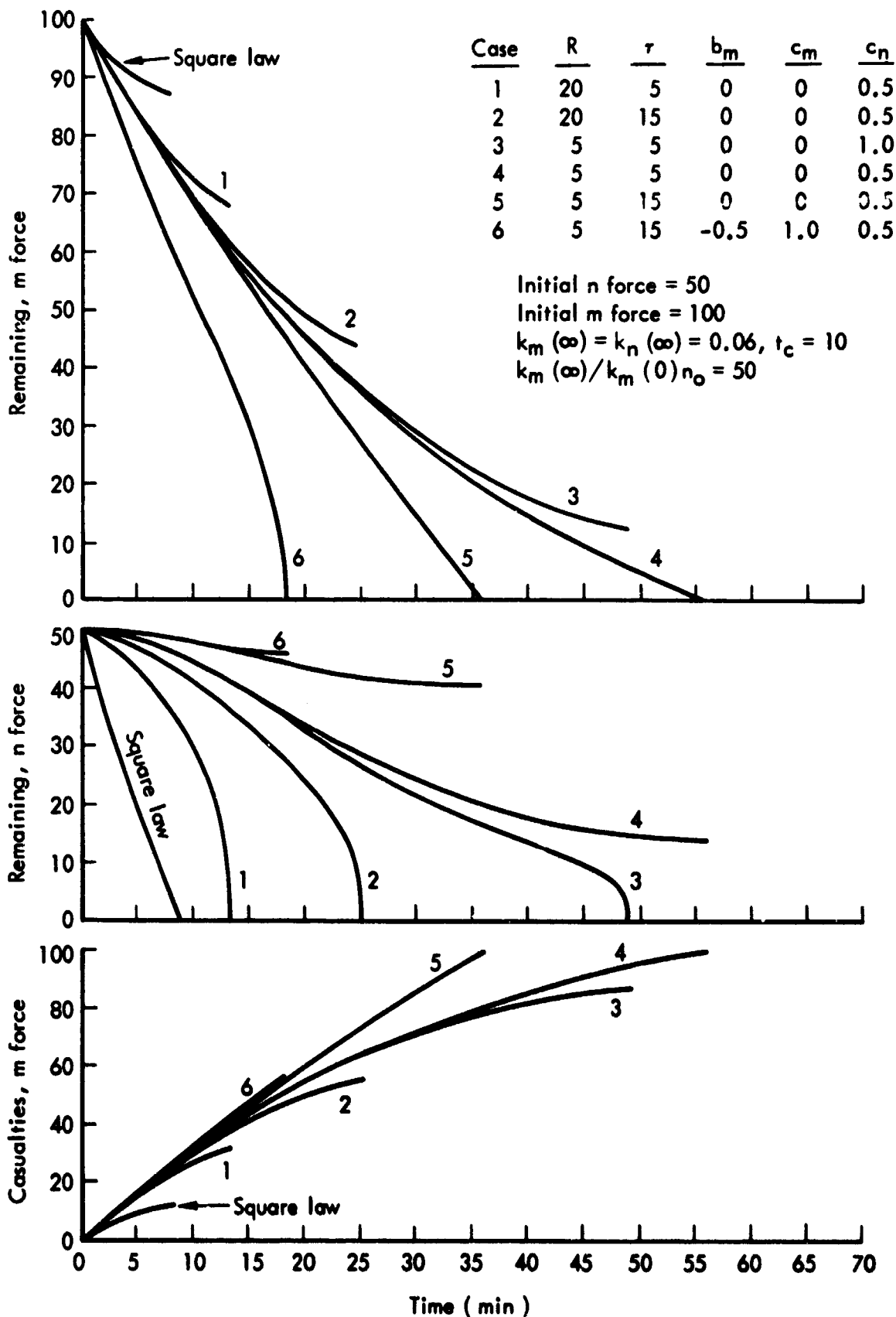


Fig.4—Ambush illustrations (no support weapons)

Table 4b

SAMPLE AMBUSH RESULTS WHERE NO SUPPORTING WEAPONS ARE USED^a
($n_o = 50, m_o = 75$)

(min)	$k_m(\omega)/k_m(o)n_o = 5000$						$k_m(\omega)/k_m(o)n_o = 50$					
	R = 2000			R = 1000			R = 500			R = 200		
	Winner	Time (min)	Time (min)	Winner	Time (min)	Time (min)	Winner	Time (min)	Time (min)	Winner	Time (min)	Time (min)
$b_m = 0, c_m = 0, c_n = 0.5$												
5	m	27.1	n	30.3	n	26.2	m	26.0	n	31.1	n	26.7
10	n	31.9	n	26.4	n	24.9	n	32.8	n	26.9	n	25.3
15	n	28.2	n	25.4	n	24.5	n	28.8	n	25.8	n	24.9
$b_m = 0, c_m = 0, c_n = 1$												
5	m	25.0	n	30.3	n	26.2	m	23.8	n	31.1	n	26.7
10	n	31.9	n	26.4	n	24.9	n	32.8	n	26.9	n	25.3
15	n	28.2	n	25.4	n	24.5	n	28.8	n	25.8	n	24.9
$b_m = -0.5, c_m = 0, c_n = 0.5$												
5	n	19.3	n	17.1	n	16.3	n	19.5	n	17.3	n	16.5
10	n	17.3	n	16.4	n	16.0	n	17.5	n	16.5	n	16.2
15	n	16.8	n	16.1	n	15.9	n	16.9	n	16.3	n	16.1
$b_m = -0.5, c_m = 0, c_n = 1$												
5	n	19.3	n	17.1	n	16.3	n	19.5	n	17.3	n	16.5
10	n	17.3	n	16.4	n	16.0	n	17.5	n	16.5	n	16.2
15	n	16.8	n	16.1	n	15.9	n	16.9	n	16.3	n	16.1
$b_m = -0.5, c_m = 1, c_n = 0.5$												
5	n	14.9	n	13.2	n	12.5	n	15.1	n	13.3	n	12.7
10	n	13.3	n	12.6	n	12.3	n	13.5	n	12.7	n	12.4
15	n	12.9	n	12.4	n	12.2	n	13.0	n	12.5	n	12.4
$b_m = -0.5, c_m = 1, c_n = 1$												
5	n	14.9	n	13.2	n	12.5	n	15.1	n	13.3	n	12.7
10	n	13.3	n	12.6	n	12.3	n	13.5	n	12.7	n	12.4
15	n	12.9	n	12.4	n	12.2	n	13.0	n	12.5	n	12.4

^a $k_n(\omega) = k_m(\omega) = 0.06; \tau_c = 10.$

Table 4c
SAMPLE AMBUSH RESULTS WHERE NO SUPPORTING WEAPONS ARE USED^a
($n_0 = 50, m_0 = 50$)

r (min)	$k_m(-)/k_m(o)n_0 = 5000$												$k_m(-)/k_m(o)n = 50$											
	R = 2000			R = 1000			R = 500			R = 20			R = 10			R = 5								
	Winner	Time (min)		Winner	Time (min)		Winner	Time (min)		Winner	Time (min)		Winner	Time (min)		Winner	Time (min)							
	$b_m = 0, c_m = 0, c_n = 0.5$																							
5	n	17.7	n	n	16.6	n	n	16.1	n	n	17.8	n	n	16.7	n	n	16.2	n						
10	n	16.7	n	n	16.1	n	n	15.9	n	n	16.8	n	n	16.2	n	n	16.0	n						
15	n	16.4	n	n	16.0	n	n	15.8	n	n	16.5	n	n	16.1	n	n	15.9	n						
$b_m = -1.0, c_m = 1, c_n = 0.5$																								
5	n	5.2	n	n	5.2	n	n	5.1	n	n	5.3	n	n	5.2	n	n	5.2	n						
10	n	5.2	n	n	5.2	n	n	5.1	n	n	5.2	n	n	5.2	n	n	5.2	n						
15	n	5.2	n	n	5.2	n	n	5.1	n	n	5.2	n	n	5.2	n	n	5.2	n						

^a $k_m(-) = k_m(m) = 0.06; t_c = 10.$

the contest. The largest contributing factor here is the number of desertions (a possible benefit to the ambushee, since the situation is hopeless even with perfect discipline).

Conceptually, a force which is ambushed by an opponent of equal numerical strength has some hope of victory if supporting weapons can be brought into the battle in timely fashion. One technique for rapid supporting-weapon response which has been widely investigated in recent years is the equipping of the lead vehicles in a column with peripheral Claymore mines and/or rapid-firing antipersonnel grenade launchers. If the column is ambushed, these devices are actuated during the early stages of conflict, creating a high volume of area fire in the general direction of the ambushers. In addition to producing casualties, such devices conceptually suppress the ambusher's fire during the stage of conflict when it is otherwise most devastating.

Illustrative examples of rapid-support weapon response by the ambushee are summarized in Table 5. As in the skirmish illustration, the $\sum E_j(t,n)W_j(t)$ are put in the form $S_c(t,n) = 0.02n, 0.2n$; however, supporting-weapon effects are permitted for only the first 2 min of the conflict. The supporting weapons are assumed to degrade the ambushers' weapon efficiency to 50 percent of its value in the absence of such weapons, as well as to produce casualties. The initial weapon-efficiency disparity is specified at 50 (a relatively favorable case for Side m), and a wide range of R , τ , and morale/discipline factors are examined.

From Table 5 it is clear that for the large majority of practical cases the introduction of the rapid-response supporting weapon prolongs the battle but does not reverse the outcome. For the most part, this holds even when the ambushers are densely concentrated (about 25 ft apart, corresponding to $S_c = 0.2 n$) and thus present a lucrative target. In a few cases, when Side m maintains perfect discipline and rapidly upgrades his weaponry from area to aimed fire, he can emerge the victor. However, examination of the casualties involved reveals that even these victories tend to be somewhat Pyrrhic. Furthermore, we have considered cases that are highly favorable to Side m, since only equal-numerical-strength cases have been examined. The successful

Table 5
SAMPLE ARMSH RESULTS WHERE SIDE m POSSESSES SUPPORT WEAPONS FOR SPECIFIED TIMES^a
($n_0 = 50, m_0 = 50$)
A. $S_c = 0, t > 2$

Combined Supporting-Weapon-Effectiveness Rate, $t = S_c / m$												
r (min)	0.02						0.2					
	R = 20			R = 5			R = 20			R = 5		
	Winner	YTime (min)	YTime (min)	Winner	YTime (min)	YTime (min)	Winner	YTime (min)	YTime (min)	Winner	YTime (min)	YTime (min)
$b = 0, c = 0, c_n = 0.5$												
5	n	20.9	n	19.0	n	18.2	n	26.4	n	30.3	n	26.3
10	n	19.2	n	18.3	n	17.9	n	31.9	n	26.5	n	25.0
15	n	18.6	n	18.0	n	17.8	n	28.3	n	25.5	n	24.6
$b = 0, c = 0, c_n = 1.0$												
5	n	20.9	n	19.0	n	18.2	n	24.0	n	30.3	n	26.3
10	n	19.2	n	18.3	n	17.9	n	31.9	n	26.5	n	25.0
15	n	18.6	n	18.0	n	17.8	n	28.3	n	25.5	n	24.6
$b = -0.5, c = 0, c_n = 1.0$												
5	n	12.9	n	12.4	n	12.2	n	19.8	n	17.5	n	16.7
10	n	12.6	n	12.2	n	12.1	n	17.7	n	16.7	n	16.3
15	n	12.3	n	12.1	n	12.0	n	17.1	n	16.5	n	16.2
B. $S_c = 0, 2 \leq t \leq 5$												
$b = 0, c = 0, c_n = 0.5$												
5	n	26.7	n	43.3	n	51.8	n	10.3	n	10.7	n	11.1
10	n	39.0	n	53.3	n	43.5	n	10.7	n	11.1	n	11.3
15	n	60.5	n	46.2	n	41.7	n	10.9	n	11.2	n	11.4
$b = 0, c = 0, c_n = 1.0$												
5	n	25.5	n	41.9	n	51.8	n	10.2	n	10.5	n	10.7
10	n	37.7	n	53.3	n	43.5	n	10.4	n	10.6	n	10.8
15	n	59.2	n	46.2	n	41.7	n	10.5	n	10.8	n	10.9
$b = -0.5, c = 0, c_n = 1.0$												
5	n	27.2	n	23.5	n	22.1	n	10.3	n	10.7	n	11.0
10	n	23.8	n	22.2	n	21.6	n	10.7	n	11.0	n	11.2
15	n	22.9	n	21.8	n	21.4	n	10.9	n	11.1	n	11.3

^a $k_m(t) = 0.06, k_n(t) = 0.06, S_c = 0, k_n(t) = 0.03, S_c \neq 0, k_n(t)/k_m(t) = 50$.

^b $n = n(t)$.

guerrilla force is usually more conservative than this and tries to outnumber the opponent in addition to ambushing him.

The final supporting-weapon case examined is summarized in the lower half of Table 5. Here, Side m not only introduces supporting weapons early in the fight ($t \leq 2$ min) but also musters heavy weapons after $t = 5$ min. (All other conditions are as before.) This is a somewhat optimistic case for defending against an ambush, but not quite as optimistic as that in which supporting weapons are available throughout the entire engagement.

Table 5 demonstrates that when the ambushed side has ample supporting weapons virtually throughout the entire engagement and the ambushers are conveniently clustered, thereby presenting lucrative targets to area weapons, it can win over an attacking force of equal size. These victories are not without significant cost to the ambushed side, which suffers up to 40 percent casualties over the range of parameters examined. The duration of the contests is not sensitive to the discipline and/or morale of Side m, or to the speed with which Side m develops aiming parity with his opponent. The reason for this becomes clear upon detailed examination of the casualties: The supporting weapons produce virtually all the Side n casualties.

When the ambushers are more suitably dispersed, victory for Side m becomes highly dependent on the effective rifle fire it can deliver and upon the discipline and/or morale it can maintain. The Side m force can win if it develops 20 percent ($R = 10$) of the enemy's weapon efficiency within 5 min, or 40 percent ($R = 20$) within 15 min, and maintains perfect discipline. These conclusions are relatively independent of the ambusher's level of discipline ($c_n = 0.5$ to 1.0). However, it becomes highly unlikely that Side m can win if its discipline is even slightly degraded ($b_m = -0.5$, $c_m = 0$). All the contests examined within the range of parameters of Table 5 were bloody and costly for both sides. This helps explain the tendency for guerrilla forces to seek a substantial force advantage, in addition to the ambush advantage, wherever possible.

Because specific ambush situations are highly nonlinear, the outcomes are difficult to determine intuitively; therefore, the descriptive

equations (even if used in order-of-magnitude fashion) can be highly useful. For example, it is not obvious that the rapid-response but nonpersistent supporting weapon which emphasizes casualty production (i.e., antipersonnel grenades) is not a very effective means of defeating a realistic ambush.

In general, the ambushee starts with very significant disadvantages in aiming capabilities, cover function, and often force size. To a certain extent, the first two of these can be minimized by proper training and aggressive tactics. The ambusher, however, can cope with aggressiveness with relative ease. He has all the advantages associated with intimate knowledge of the terrain and can allow for numerous maneuver and reinforcement contingencies.

The ambushee can make a very significant breakthrough if the attacker's capability is reduced to the employment of area fire early in the conflict. Conceptually, this would provide the time necessary for the defense to maneuver and simultaneously keep its casualties within manageable proportions. One technique which might accomplish this is the substitution of CS tear-gas grenades for the rapid-reaction fragmentation grenades discussed above. In addition to emphasizing fire-suppression, the effects of the CS would be relatively persistent, thus reducing the logistical burden. An interesting variation of the idea is to use CS grenades initially, creating the persistent-suppression effect, and then shift to fragmentation grenades.

In summary of the discussion of small-force ambushes, the following can be shown analytically:

1. In the absence of supporting weapons, ambushes can be successful against forces which are numerically twice as large as the ambusher's force, provided the ambushee has less than perfect discipline and/or is sluggish in attaining aiming parity with his opponent. Even when they are ultimately unsuccessful, ambushes are quite costly in terms of casualties for the ambushee. This suggests that a useful tactic for the smaller force is to initiate an ambush with the intention of disengaging when the defenders have reached near-parity in aiming ability and cover.

2. In the absence of supporting weapons, well-prepared and executed ambushes are usually successful against forces numerically 50 percent larger than the ambusher.

3. Supporting weapons brought into play by the ambushee and expended in the early stages of the conflict do not appear to be an effective means of reversing the tide of battle.

4. When supporting weapons are available to the ambushee for virtually the entire battle, the equal-force ambush fails if the attackers are closely clustered and thus offer lucrative targets to area weapons. If the ambushers are suitably dispersed, the ambushee can win, provided his discipline is perfect and he rapidly approaches aiming parity with his enemy. These possibilities of ambush failure in the equal-force case probably explain why successful guerrilla bands attempt to seek a force advantage in addition to the surprise advantages offered by an ambush.

5. A useful technique for countering ambushes is to utilize rapid-response fire-suppression weapons (e.g., CS grenades) to distract the ambusher and degrade his fire to area fire.

VI. SIEGE

Sieges can be divided into two stages: (1) an initial "softening-up" phase where support weapons are of prime importance (the riflemen are generally out of range), and (2) an assault stage where the offensive artillery barrage must of necessity be lifted. Of course, not all sieges involve both phases; some are abandoned upon completion of the first, and others begin with the second phase, omitting the first phase entirely. Some of the criteria used by the attacker in deciding whether or not to undertake two phases of siege are illuminated by the analysis which follows.

The attacker has two major objectives in mounting the first stage of a siege: first, to reduce the defender's supporting weapons (if they exist) to a negligible level, and second, to soften the defensive perimeter to permit an assault. To ascertain whether his objectives are achievable, the attacker makes a series of decisions, more or less as follows:

1. If the defender has no immediate supporting weapons, the attacker decides whether or not the benefits of softening the perimeter outweigh the loss of surprise. If a surprise attack is decided upon, the assault is undertaken directly. If the attacker opts for softening the perimeter, the barrage and subsequent assault must be timed so that the defender has little opportunity to bring in supporting aircraft or remotely located artillery while the attack is in progress.

2. If the defender has immediate supporting weapons (artillery based within the compound, for example), the attacker attempts to gauge the number of barrage rounds necessary to neutralize the defending support. His decision is heavily influenced by the level of his knowledge concerning their location. The attacker also must be concerned with the defensive perimeter. He therefore attacks it with his support weapons either simultaneously or subsequently.

3. The attacker undertakes the assault if he estimates that the defensive supporting weapons have been neutralized, the defensive perimeter is negotiable, and the attack can be completed before heavy defensive supporting fire arises. Considering the hit-and-run nature of

Phase II guerrilla doctrine, the attacker will abandon the siege if any of these criteria are violated.

The softening-up phase is, therefore, either a one- or two-sided artillery duel with men and/or support weapons as targets. The equations applicable to manpower attrition are

$$\left. \begin{aligned} \frac{dm}{dt} &= -(1 - b_m) \sum_1^i E_1(m) W_1 \\ \frac{dn}{dt} &= -(1 - b_n) \sum_j^j E_j(n) W_j(t) \end{aligned} \right\} \quad (18)$$

Assuming that weapons of a single type are used, and letting the targets on both sides be randomly dispersed, the $\sum E \cdot W$ can be put in the form of Eq. (8). The first part of Eq. (18) is then solved directly, yielding

$$\ln \frac{m_0}{m_1} = (1 - b_m) \frac{k_n A_L}{A_m} (t_1 - t_0) \quad (19)$$

where the numerical subscripts represent conditions at the beginning and end of the first phase. Equation (19) can be used to estimate Side m casualties in the defensive perimeter, letting A_m be the area within which such troops are deployed.

We note in the second part of Eq. (18) that the variable $W_j = W_j(t)$. This permits the attackers to engage the defensive artillery directly and thus reduce the defensive-artillery strength during the course of battle. This asymmetric consignment of offensive capability to attack the opponent's supporting weapons, combined with the denial of a similar capability for defensive weapons, is a (not unusual) concession to the side initiating the action and is most easily associated with better preparation for the battle. In effect, the attackers are given some intelligence regarding the location of defensive artillery, and the defense is denied similar intelligence about their attackers.

Reflecting the foregoing, and delaying the use of defensive artillery for Δ min, the second part of Eq. (18) can be solved to yield

$$\ln \frac{n_0}{n_1} = \frac{(1 - b_n) A_{Ln}}{A_n} \int_{t_0 + \Delta}^{t_1} \bar{k}_m(t) dt \quad (t_1 \geq t_0 + \Delta) \quad (20)$$

As an approximation, $\bar{k}_m(t)$ may be described by a relation of the form

$$\bar{k}_m(t) = \bar{k}_{m_0} e^{-\delta t} \quad (21)$$

where δ is a positive constant reflecting the effectiveness of the Side n attack. Substituting in Eq. (20), and integrating,

$$\ln \frac{n_0}{n_1} = \frac{(1 - b_n) A_{Ln} \bar{k}_{m_0}}{A_n \delta} \left(e^{-\delta(t_0 + \Delta)} - e^{-\delta t_1} \right) \quad (22)$$

The attacker bases the timing of the assault largely on Eqs. (19) and (21). Figure 5 plots both of these equations parametrically (they are of the same form). In many instances it is reasonable for the attacker to expect to complete his barrage before defensive artillery can even be brought into the battle ($t_1 - t_0 \leq \Delta$). In any event, the assault will not be undertaken if the defense has substantive artillery remaining within the assault area. The $K = \delta$ curves of Fig. 5 can be used to estimate whether or not that condition exists. Similarly, the $K = (1 - b_n) A_{Ln} \bar{k}_{m_0} / A_n$ curves can be used to estimate the degree to which the defensive perimeter has been softened.

The casualties on the attacking side (Side n, Eq. (22)) are represented in Fig. 6. For compactness, it is necessary to plot these results as a function of dimensionless time, δt . The dimensionless constant $G = (1 - b_n) A_{Ln} \bar{k}_{m_0} / A_n \delta$ (from Eq. (22)) is useful as an

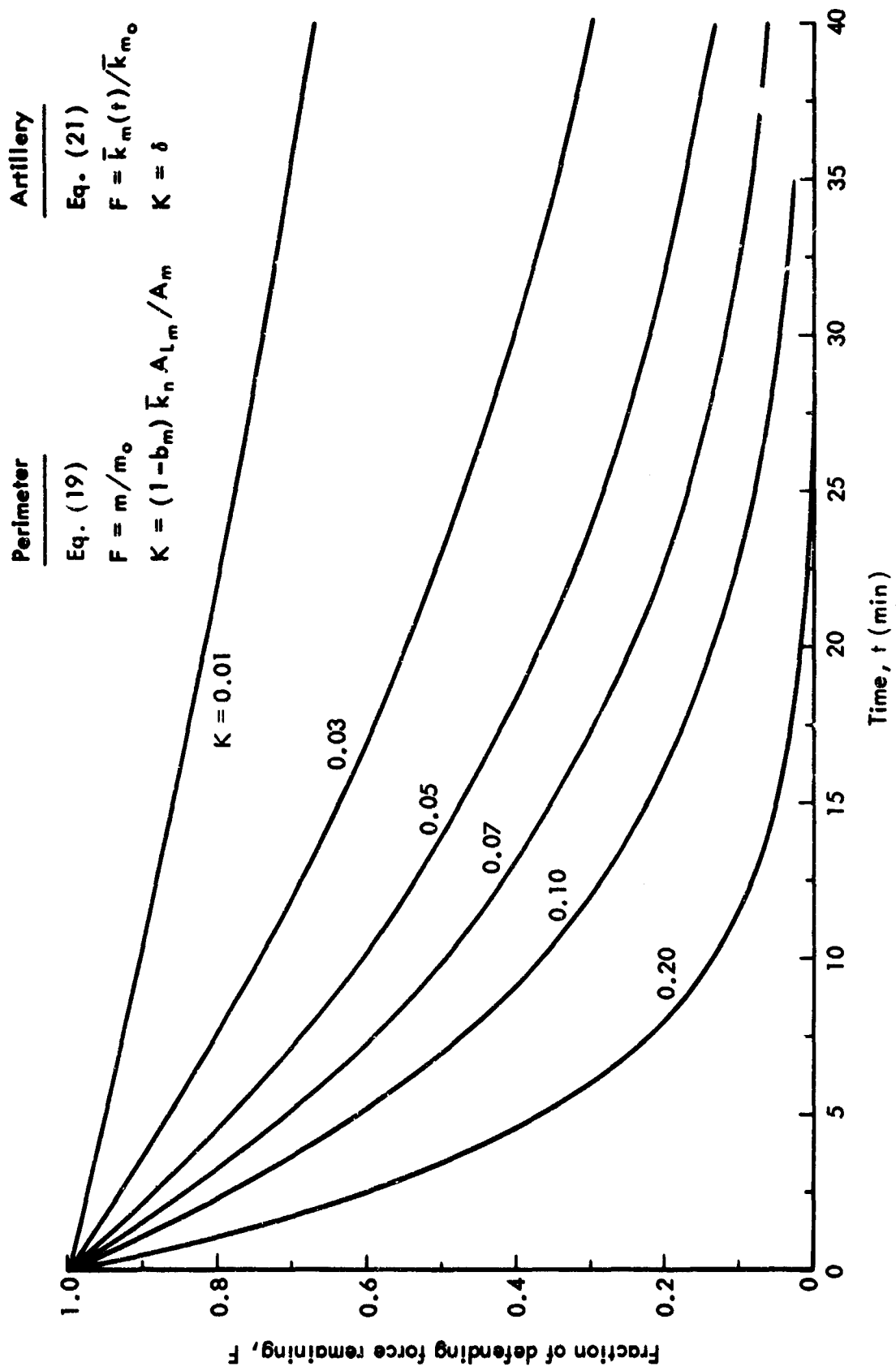


Fig. 5—Remaining defense force in Phase I siege

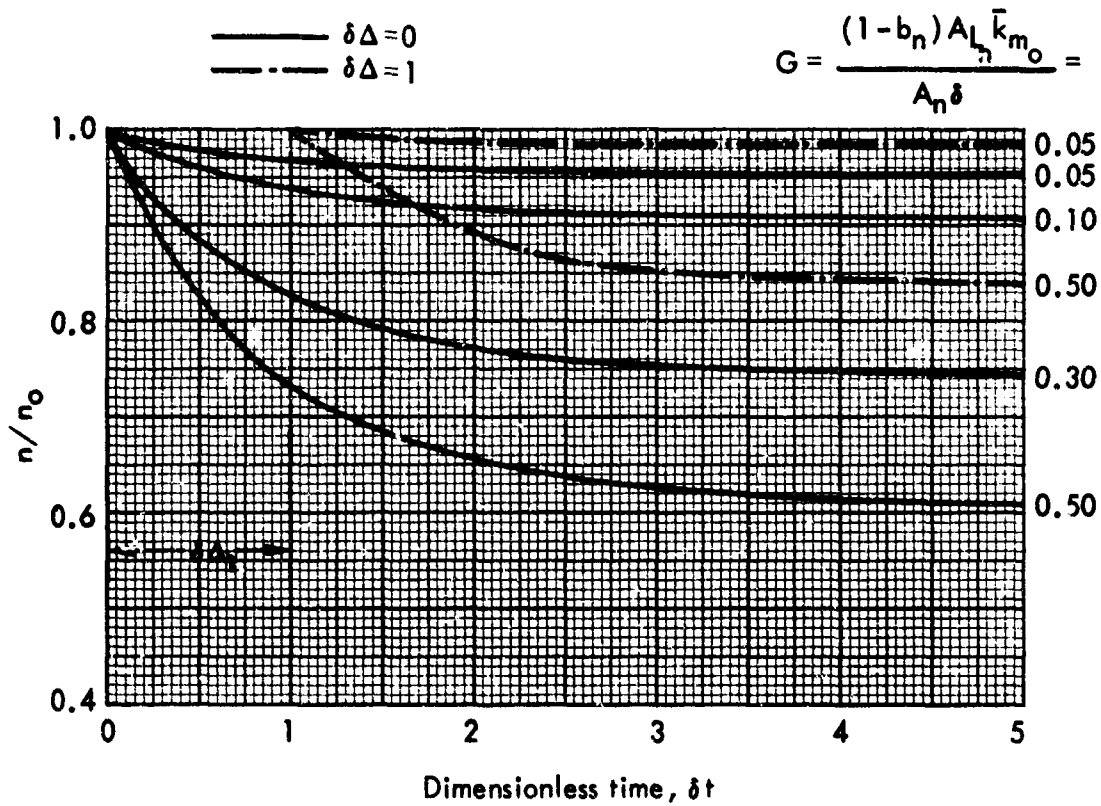


Fig.6—Remaining attacking force in Phase I siege

independent parameter. Values are shown for the dimensionless delay times, $\delta\Delta$, of 0 and 1.

In general, the casualties on the attacking side during Phase I of a siege are relatively low, and significant delays on the part of the defensive artillery make them much lower. For example, if the parameters used for the supporting weapon-skirmish illustrations are applied here ($A_{Ln} = 600 \text{ ft}^2$; $\bar{k}_{m_0} = 15 \text{ rds/min}$; $A_n = 500,000 \text{ ft}^2$), the results for various δ 's and $t = 10 \text{ min}$ are as shown in Table 6.

Table 6
PHASE I OF SIEGE AFTER 10 MIN

δ	G	δt	$\frac{n}{n_0}$	$\frac{\bar{k}_m(t)}{\bar{k}_{m_0}}$	$\frac{m}{m_0}$
0.05	0.36	0.5	0.87	0.6	0.83
0.10	0.18	1.0	0.90	0.28	0.83
0.20	0.09	2.0	0.93	0.13	0.83

Corresponding values of the defensive attrition (support weapons and manpower, respectively) are given in the last two columns of Table 6. Thus, after the first 10 min of Phase I siege, the defensive artillery is substantially reduced, while for all values of δ , the defensive manpower losses are greater than those of the attackers.* If the use of defensive artillery is delayed 5 min, the remaining attacking forces are increased to 0.94 for $\delta = 0.05$, 0.96 for $\delta = 0.10$, and 0.98 for $\delta = 0.20$.

Turning now to Phase II of the siege, an assault against a constant area defense can be described by the mixed linear-square law (following Brackney's derivation⁽⁴⁾):

*We have considered a very optimistic case for the defense, $A_m = 500,000 \text{ ft}^2$. It is not unreasonable to expect the defense to be clustered in 1/10 this area on many occasions, which results in a corresponding m/m_0 of 0.19.

$$\left. \begin{aligned} \frac{dm}{dt} &= -\left(1 - b_m\right) P_{K_n}^{nm} / \gamma_n A_m \\ \frac{dn}{dt} &= -\left(1 - b_n\right) P_{K_m}^{m} / t_{f_m} \end{aligned} \right\} \quad (23)$$

where $\gamma_n A_m$ is the average time for an assault troop to acquire a defensive target, and t_{f_m} is the average time for the defense to fire after the acquisition of a target.

Equations (23) have the time-independent (indifference) solution

$$m_1 - m = \frac{\left(1 - b_m\right) P_{K_n}^{nm}}{\left(1 - b_n\right) P_{K_m}^{m}} \left(\frac{t_{f_m}}{2 \gamma_n A_m} \right) (n_1^2 - n^2) \quad (24)$$

and a time-dependent general solution developed by Deitchman.⁽⁵⁾ This demonstrates the advantage to the defense of the use of rapid-firing weapons (t_{f_m} small) and the disadvantage to the defense of concentrated forces (A_m small), in addition to the usual dependencies on weapon efficiencies and discipline and/or morale.

Equation (24) conceptually describes the assault situation up to the time the defensive perimeter is overrun, or until a counterattack is launched. From that point to completion of the battle or disengagement, the form of Lanchester's linear law for hand-to-hand combat (see footnote on p. 5) is applicable.

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10. ABSTRACT This study develops deterministic forms of Lanchester-type equations modeling small-force guerrilla engagements that are typical of the early stages of insurgency. Three types of appropriate military activity are identified and treated mathematically: skirmish, ambush, and siege. In addition to the usual treatment of casualty production, these models include the effects of troop morale, troop captures, and supporting weapons. Since the descriptive equations are not amenable to closed-form solution, numerical results for illustrative parameters are obtained.		11. KEY WORDS Limited warfare Models Equations War gaming Counterinsurgency	